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Singularities of integrable geodesic flows on multidimensional torus and sphere

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Abstract

The main purpose of this work is to give a complete topological description of (nonlocal) singularities of geodesic flows on multidimensional ellipsoids. These geodesic flows were studied before by many authors, but the structure of singularities was not known except for some cases of codimension 1. We consider some other geodesic flows as well, namely that of Liouville metrics on tori, and perturbations of ellipsoids' metric on spheres.

Keywords: Ellipsoid; Integrable geodesic flow; Singularity
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1. Introduction and statement of the results

The recent study of phase topology (i.e. topology of the phase space together with the dynamical system) in classical mechanics owes much to Smale's program [Sm]. One of the main ideas is to use the bifurcation diagrams. Notice that singularities play a very important role in this program.

For integrable Hamiltonian systems, Fomenko created a Morse-like topological theory, using Smale's and others' ideas (see, e.g. [Fo]). His topological invariants are essentially based on codimension 1 singularities. Thus Fomenko's theory works well for systems with two degrees of freedom, where on regular isoenergy surfaces all singularities are of codimension 1.

For systems with more degrees of freedom (or even systems with two degrees of freedom if we do not restrict to an isoenergy submanifold), singularities of higher codimension naturally occur. Thus to have a good Morse-like theory in this case, we first have to understand the structure of typical singularities.

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In [Z1,Z2] we studied the topological structure of general nondegenerate singularities of any codimension of integrable Hamiltonian systems. We recall some necessary notions from there.

Consider an integrable Hamiltonian system $v = X_H$ generated by a Hamiltonian H on a symplectic $2n$-dimensional manifold (M^{2n}, ω). There exist n commuting first integrals $F_1 = H, F_2, \dots, F_n$ for this system, and we will assume that F_i can be chosen so that they are independent almost everywhere, and their common level sets are compact. By Arnold–Liouville theorem, these common level sets are unions of Liouville tori wherever nonsingular. So we have an associated singular foliation of M by Liouville tori. In case H is nonresonant, as we will always assume, this foliation depends only on H and not on the additional first integrals. The orbit space, or base space, of this foliation will be denoted by $C(M, v)$ and called the *orbit space of* v (cf. [Z2]). A point $x \in M^{2n}$ is called singular of *corank* k if the rank of the differential of the moment map $\boldsymbol{F}$, denoted by $D\boldsymbol{F}$, is equal to $(n - r)$ at x. It is called a *nondegenerate* singular point if its transversal linearization is nondegenerate (see, e.g. [El,DM,De,Z2]). A singular leaf, denoted by N, of the Lagrangian foliation by Liouville tori, is called *nondegenerate* if all of its points are either nonsingular or nondegenerate singular. The *codimension of* N is the maximal corank of the points in N. A (nonlocal, or semi-local) *singularity* is a germ of the Lagrangian foliation near a singular leaf N (in topological sense). It can be considered as a small tubular neighborhood $\mathcal{U}(N)$ of N with the singular foliation by Liouville tori in it, and such that $\mathcal{U}(N)$ is saturated by the leaves of the foliation in M^{2n}. If the codimension of N is k, we say that $\mathcal{U}(N)$ is a (k, n)-singularity. Two singularities $\mathcal{U}(N_1)$ and $\mathcal{U}(N_2)$, of type (k, n_1) and (k, n_2) respectively $(n_1 \le n_2)$, are called *topologically equivalent* if $\mathcal{U}(N_2)$ is homeomorphic, by a foliation-preserving homeomorphism, to the direct product $\mathcal{U}(N_1) \times \mathcal{U}(N_3)$, where $\mathcal{U}(N_3)$ is a tubular neighborhood of a regular Liouville torus of an integrable system with $n_2 - n_1$ degrees of freedom. A nondegenerate (k, n)-singularity $\mathcal{U}(N)$ is called *stable* if the following conditions are satisfied:

(1) There is a diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ such that by taking the composition $\boldsymbol{F}' = \Phi \circ \boldsymbol{F}$ of the moment map with this diffeomorphism, we have that, under the new moment map $\boldsymbol{F}'$ restricted to $\mathcal{U}(N)$, $\boldsymbol{F}' : \mathcal{U}(N) \to \mathbb{R}^n = \{(x_1, \dots, x_n)\}$, the singular value set of this map in $\mathbb{R}^n$ is contained in the union of the hyperplanes $\bigcup_{i=1}^n \{x_i = 0\}$ (and N is mapped to the origin).

(2) In N all the closed orbits of the Poisson action (generated by $\boldsymbol{F}$) are tori of the same dimension $(n - k)$.

Recall also from [Z2] that condition 2 is automatically satisfied if Williamson's type of N does not contain focus–focus components (and it is the case with our geodesic flows). Furthermore, all stable nondegenerate singularities are decomposable to the direct product of simplest codimension 1 or codimension 2 focus–focus components, after a finite covering.

In this work we deal with geodesic flows of Liouville metrics on the torus $\boldsymbol{T}^n$ and standard metrics on ellipsoids $\boldsymbol{S}^n$ (and their perturbations). It is known that these geodesic flows are integrable, and they are important because many problems in classical mechanics can be reduced to these systems (see, e.g. [AKN]).

From codimension 1 singularities (i.e. letter-atoms in the terminology of Fomenko) we will need the following: $\mathcal{A}$, $\mathcal{A}^*$, $\mathcal{B}$, $\mathcal{P}_k$, $\mathcal{V}_k$. The meaning of these letters are as follows: $\mathcal{A}$ is an elliptic singularity. $\mathcal{A}^*$ is a simplest normally nonorientable hyperbolic singularity. $\mathcal{B}$ is a simplest (normally orientable) singularity. $\mathcal{P}_k$, $k \geq 2$, is the "cyclic" hyperbolic singularity: it has k vertices situated in a cyclic order in a plane, and every vertex has two common edges with the next vertex and two common edges with the previous one. $\mathcal{V}_k$, $k \geq 2$, is the "chain" hyperbolic singularity: it has k ordered vertices, and every vertex is connected to the previous and the next vertices by 2 edges each as in case of $\mathcal{P}_k$ except that the first vertex has a loop (since its previous does not exist) and the same thing holds for the last vertex. See [PZ,PSZ] for the picture of codimension 1 singularities described above. By convention, the singularity $\mathcal{P}_2$ will be denoted $\mathcal{C}_2$.

We can now formulate the main results.

Let $\mathrm{d}s^2 = \sum g_i(q_i) \sum \mathrm{d}q_i^2$ be a Liouville metric on the torus $\boldsymbol{T}^n$ with periodic coordinates q_i, where g_i – smooth positive functions. It is known (see, e.g. [AKN]) that the geodesic flow of this metric is an integrable Hamiltonian system on $T^*\boldsymbol{T}^n$ (with the standard symplectic form $\omega = \sum \mathrm{d}p_i \wedge \mathrm{d}q_i$ and the corresponding Hamiltonian $H = \sum p_i^2 / \sum g_i(q_i)$).

Theorem 1. *Assume that g_i are Morse functions. Then for the above Liouville geodesic flow we have*:

(a) *It is a strongly nondegenerate system on $T^*\boldsymbol{T}^n \backslash \boldsymbol{T}_0^n$ (where $\boldsymbol{T}_0^n$ is the zero section), and nonresonant for almost any choice of g_i (almost any means a dense open subset).*

(b) *Any codimension one singularity of this system is of the type $\mathcal{A}$, $\mathcal{B}$, $\mathcal{P}_k$ or $\mathcal{V}_k$.*

(c) *Any higher codimension singularity is the direct product (topologically) of codimension 1 singularities of the above types $\mathcal{A}$, $\mathcal{B}$, $\mathcal{P}_k$,$\mathcal{V}_k$. Conversely, any direct product of the above types can be realized (with an appropriate choice of functions g_i of the metric).*

Remark. If we call the (unordered) set $(S(g_1), \dots, S(g_n))$ the *code* of the geodesic flow, where $S(g_i)$ is the code of the function g_i defined as in [PZ,PSZ], then as in these papers, two Liouville metrics yield topologically equivalent Hamiltonian systems if and only if their codes are equivalent. Recall that two integrable Hamiltonian systems are called *topologically equivalent* if the corresponding singular Lagrangian foliations are homeomorphic.

Consider now an n-dimensional ellipsoid

$$\boldsymbol{S}^n = \left\{ \sum_1^{n+1} x_i^2 / a_i = 1 \right\} \tag{1}$$

in the Euclidean space $\mathbb{R}^{n+1}$ with the induced metric. The integrability of the geodesic flow on such an ellipsoid was proved by Jacobi in the last century. We will always assume the following general position condition:

$$0 < a_1 < a_2 < \cdots < a_{n+1}. \tag{2}$$

Theorem 2. *The geodesic flow on the above ellipsoid S^n, considered as a Hamiltonian system on $T^n S^n \backslash S^n_0$, is strongly nondegenerate and has the following properties:*

(a) *All codimension 1 singularities are $\mathcal{A}, \mathcal{C}_2 (n = 2); \mathcal{A}, \mathcal{B}, \mathcal{C}_2 (n = 3); \mathcal{A}, \mathcal{A}^*, \mathcal{B}, \mathcal{C}_2$ $(n \geq 4)$.*

(b) *Any higher codimension singularity is either equivalent to one of the following singularities or to a direct product of some of them: $\mathcal{A}, \mathcal{A}^*, \mathcal{B}, \mathcal{C}_2, El^i_j$ $(i = 1, 2, 3; j = 1, 2, \ldots)$. Here El^i_j is a stable nondegenerate codimension k singularity. Their description is given in the next theorem.*

(c) *There are exactly 2n stable closed orbits of the system, i.e. exactly n stable closed geodesics on S^n (not counting the orientation). They are given by the equations*

$$\left\{ x = (x_i) \;\middle|\; \sum_{1 \leq i \leq n+1,\, i \neq k,\, k+1} x_i^2 = 0 \right\} \quad (k = 1, \ldots, n). \tag{3}$$

(d) *Geodesic flows of any two ellipsoids with the same dimension (and in general position) are topologically equivalent.*

In Section 3, we will also consider perturbations of the ellipsoid's metric in some integrable class: the associated geodesic flows are still integrable. These new geodesic flows can also admit $\mathcal{V}_k$ in the components of their singularities.

Since all the singularities El^i_j in Theorem 2 are stable nondegenerate, according to [Z2] they must be decomposable, after a finite covering, to a direct product of simplest singularities. Indeed, we have:

Theorem 3. *Any of the singularities El^i_j in Theorem 2 admits a 2^s-sheet regular covering (for some natural number s), which is a direct product of singularities of types $\mathcal{B}$ and $\mathcal{C}_2$.*

A more detailed description of El^i_j is given in Section 3.

The proof of the above theorems is given in the subsequent sections. The approach that we use is the classical method of separation of variables (MSV), in terms of the so-called Stäckel systems. We hope that other systems, which are integrable by MSV, can be dealt with in a similar way.

2. Liouville geodesic flow on torus

We recall some facts about Stäckel systems (see, e.g. [AKN,Wo]). Let $\mathcal{O}$ be an open subset of the standard symplectic space $\mathbb{R}^{2n}(p_i, q_i)$. A *Stäckel system* in $\mathcal{O}$ is a Hamiltonian system $v = X_H$ in $\mathcal{O}$ given by a Hamiltonian function H of the type

$$H = \sum_{s=1}^{n} \frac{\Phi_{1s}(q) f_s(p_s, q_s)}{\Phi(q)}, \tag{4}$$

where f_s are smooth functions on p_s, q_s; $\Phi(q) = \det(\phi_{ij}(q_j))^n_{i,j=1}$, where $\phi_{ij}(q_j)$ – smooth functions depending only on q_j, and $\Phi_{kl}(q)$ is the algebraic complement to element ϕ_{kl} in the matrix (ϕ_{ij}). Here we suppose that $\Phi(q)$ is nonzero in $\mathcal{O}$.

It is known that this system is integrable with the aid of the following commuting first integrals:

$$F_i = \sum_{s=1}^{n} \frac{\Phi_{is} f_s}{\Phi} \quad (i = 1, \ldots, n, F_1 = H). \tag{5}$$

We are interested in the points where the first integrals (F_i) are dependent, i.e. in the singular points. Since the matrix (ψ_{ij}), where from now on we will denote $\psi_{ij} = \phi_{ji}$, is invertible by assumption, the set of singular points is given by the following lemma.

Lemma 1. *The following equality holds:*

$$(\psi_{ij}) \left(\frac{\partial \boldsymbol{F}}{\partial p}, \frac{\partial \boldsymbol{F}}{\partial q} \right) = \begin{pmatrix} \dfrac{\partial f_1}{\partial p_1} & 0 & 0 & G_1 & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & \dfrac{\partial f_n}{\partial p_n} & 0 & 0 & G_n \end{pmatrix} \tag{6}$$

(the right-hand side consists of two diagonal matrices), where

$$G_i = \frac{\partial f_i}{\partial q_i} - \sum_{k=1}^{n} F_k \frac{\partial \psi_{ik}}{\partial q_i}. \tag{7}$$

Proof. From Eq. (5) for integrals F_i we have

$$(\psi_{ij}) \boldsymbol{F} = f, \tag{8}$$

where $\boldsymbol{F} = (F_i)$, $f = (f_i)$. As a consequence

$$\frac{\partial}{\partial p}((\psi_{ij})\boldsymbol{F}) = \frac{\partial f}{\partial p}. \tag{9}$$

Since (ψ_{ij}) does not depend on p, and f_i depends only on p_i, q_i for every i, Eq. (9) means that

$$(\psi_{ij}) \frac{\partial \boldsymbol{F}}{\partial p} = \mathrm{diag}\left(\frac{\partial f_i}{\partial p_i} \right). \tag{10}$$

Analogously, since every function ψ_{ij} depends only on q_i, we have

$$\frac{\partial f}{\partial q} = \frac{\partial}{\partial q}((\psi_{ij})\boldsymbol{F}) = (\psi_{ij}) \frac{\partial \boldsymbol{F}}{\partial q} + \mathrm{diag}\left(\sum_{k=1}^{n} F_k \frac{\partial \psi_{ik}}{\partial q_i} \right). \tag{11}$$

In other words, $(\psi_{ij}) \partial \boldsymbol{F} / \partial q = \mathrm{diag}(G_i)$. □

Corollary 1. *A point $x \in \mathcal{O}$ is nonsingular (with respect to the system of integrals $\boldsymbol{F} = (F_i)$) if and only if at this point we have $\prod_{i=1}^{n}((\partial f_i / \partial p_i)^2 + G_i^2) \neq 0$. Moreover, if x is* *a singular point then the corank of x is equal to the number of indices i for which the following equality holds*:

$$\frac{\partial f_i}{\partial p_i} = G_i = 0. \qquad \square \tag{12}$$

We will show that Liouville geodesic flows are particular cases of Stäckel systems. The proof of the following lemma is straightforward.

Lemma 2. *The Hamiltonian*

$$H = \frac{\sum_{i=1}^{n} f_i(p_i, q_i)}{\sum_{i=1}^{n} g_i(q_i)}, \tag{13}$$

where $g_i(q_i)$ are the smooth positive functions, has the Stäckel type (4) *with*

$$(\phi_{ij}) = \begin{pmatrix} g_1 & g_2 & \dots & g_{n-1} & g_n \\ -1 & 1 & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -1 & 1 \end{pmatrix}. \qquad \square \tag{14}$$

Now look at a Liouville geodesic flow on T^n. Its Hamiltonian $H = \sum p_i^2 / \sum g_i(q_i)$ has the type (13) with $f_i(p_i, q_i) = p_i^2$. Thus we can consider our geodesic flow as a Stäckel system with $f_i(p_i, q_i) = p_i^2$ and (ϕ_{ij}) given by Eq. (14). The singularity condition (12) (for the Poisson action generated by the system of first integrals F given in Eq. (5)) in our case can be rewritten as follows:

$$p_i = \mathrm{d}g_i/\mathrm{d}q_i = 0. \tag{15}$$

Eq. (8) has the following form:

$$\begin{aligned} &g_1 H - F_2 = p_1^2, \\ &g_2 H + F_2 - F_3 = p_2^2, \\ &\vdots \\ &g_{n-1} H + F_{n-1} - F_n = p_{n-1}^2, \\ &g_n H + F_n = p_n^2. \end{aligned} \tag{16}$$

In other terms

$$g_i H + C_i = p_i^2, \quad i = 1, \dots, n, \tag{17}$$

where

$$C_1 = -F_2, \qquad C_2 = F_2 - F_3, \dots, C_{n-1} = F_{n-1} - F_n, \qquad C_n = F_n. \tag{18}$$

Thus the Lagrangian foliation by Liouville tori is given by the equations

$$H = \text{const}, \qquad C_i = \text{const} \quad (i = 1, \dots, n) \tag{19}$$

in the cotangent bundle $T^*\boldsymbol{T}^n$.

In the following lemma, instead of the original Hamiltonian flow we will consider the Poisson action generated by the moment map $\boldsymbol{F} = (F_i)$, given by (16).

Lemma 3. *Let $g_i(q_i)$ be Morse functions as before. Then a point $x = (p, q) \in T^*\boldsymbol{T}^n \backslash \boldsymbol{T}^n_0$ is a regular point of the system (i.e. the Poisson action of the moment map $\boldsymbol{F}$) if and only if at this point we have*

$$\prod_{i-1}^{n} (p_i^2 + (\mathrm{d}g_i/\mathrm{d}q_i)^2) \neq 0. \tag{20}$$

A point x is singular of corank k if and only if there are exactly k indices i such that at x we have $p_i = \mathrm{d}g_i/\mathrm{d}q_i = 0$.

Proof. Recall that the level sets of the moment map are given by Eq. (17). Thus each level set is the direct product of the curves $\{g_i H + C_i = p_i^2\}$ on the cylinders $T^*T^1(p_i, q_i)$. If at a point x we have $p_i = \mathrm{d}g_i/\mathrm{d}q_i = 0$ (for some i), then the curve $\{g_i H + C_i = p_i^2\}$ is singular at x (i.e. at $(p_i(x), q_i(x))$. (More precisely, x will be either an isolated point or a point of transversal self-intersection.) Lemma 3 follows easily from this observation. □

In order to see the nonresonance of the original Hamiltonian system, we have to look at a system of action coordinates. Recall [Z1] that a system of action coordinates (for an integrable Hamiltonian system) is a system of action components of the system of action–angle coordinates. Also, if γ is a 1-cycle on a Liouville torus and we extend it to nearby tori by homotopy, and α is a 1-form such that $\mathrm{d}\alpha$ equals to the symplectic form, then an action function (i.e. a function in a system of action coordinates) can be given by Arnold's formula $A = \int_\gamma \alpha$. Applying the above formula to our case, we obtain a system of action functions $A = (A_i)$ $(i = 1, \dots, n)$ with

$$A_i = \int \sqrt{g_i H + C_i}\, \mathrm{d}q_i \tag{21}$$

(the integral is taken on a connected component of the set $\{g_i H + C_i \geq 0\}$).

Of course, A is also a full system of commuting first integrals for our flow (they are constant on Liouville tori). So H can be written as a function of (A_i). Fix a value of H. Then when C_i vary, A will vary according to (21) so that to draw a hypersurface in the space $\mathbb{R}^n = \{(A_i)\}$, which is in fact a level set of H. There is only one relation between C_i: $\sum C_i = 0$. It follows that

$$\left(\frac{\partial H}{\partial A_i}\right) \propto \left(\frac{1}{\partial A_i/\partial C_i}\right).$$

Thus to show that Hamiltonian H is nonresonant, it is enough to compute $\partial A_i/\partial C_i$ and show that these values are rationally independent (for a subset of full measure of the orbit space).

For convenience fix $H = 1$ (by homogeneity of geodesic flows, we can do so). Fix an index i. Then A_i is simply the action function for a system with one degree of freedom and with the potential equal to g_i. Then some simple arguments show that for almost any g_i, the function $\partial A_i/\partial C_i$ (as a function on C_i) is nonconstant: its derivative is nonzero almost everywhere. It follows easily that for almost any choice of (g_i) the family $(\partial A_i/\partial C_i)_{i=1}^n$ is rationally independent almost everywhere. In other words, we have:

Lemma 4. *For almost any choice of Morse functions g_i, the Liouville geodesic flow on the n-dimensional torus is nonresonant.* □

Remark. It is possible to choose g_i (every g_i is of the type const $- q_i^2$ locally somewhere, say), so that the Liouville geodesic flow is resonant in some domain of $T^*\boldsymbol{T}^n$.

Proof of Theorem 1. It follows easily from Lemmas 3 and 4, by the same standard arguments as in [PZ,PSZ].

3. Geodesic flow on the ellipsoid

Since the ellipsoid $S^n = \{\sum_{i=1}^{n_1} x_i^2/a_i = 1\}$, $a_1 < a_2 < \cdots < a_{n+1}$, does not have global systems of coordinates, it is difficult to write down directly the Hamiltonian and first integrals for its geodesic flow. Our idea is to make use of a ramified covering $\boldsymbol{T}^n \to \boldsymbol{S}^n$ to pull the metric on $\boldsymbol{T}^n$ and then apply the results of the previous section.

Following Jacobi, we will use the so-called *elliptic coordinates* on S^n, $\lambda_1, \ldots, \lambda_n$, which are nonzero solutions of the equation

$$\sum_1^{n+1} \frac{x_i^2}{a_i - \lambda} = 1 \tag{22}$$

and with the property that $a_1 < \lambda_1 < a_2 < \lambda_2 < \cdots < \lambda_n < a_{n+1}$.

The system of elliptic coordinates (λ_i) is regular at any point $x \in S^n$ for which $x_i \neq 0$ ($i = 1, \ldots, n+1$). However, this system is singular on subellipsoids

$$S_k^{n-1} = \{x = (x_i) \in S^n \mid x_k = 0\} \quad (k = 1, \ldots, n+1). \tag{23}$$

We consider how the functions λ_i behave near that subellipsoids.

First note that for every i the critical point set of the function λ_i is contained in the union $S_k^{n-1} \cup S_{k+1}^{n-1}$.

From $\sum x_i^2/a_i = \sum x_i^2/(a_i - \lambda_i) = 1$ we have

$$\frac{x_k^2}{a_k(\lambda_k - a_k)} = \sum_{i \neq k} \frac{x_i^2}{a_i(a_i - \lambda_k)} \stackrel{\text{def}}{=} g_k(x, \lambda_k). \tag{24}$$

For a fixed point $x = (x_i) \in S^n$ with x_k near to 0 (it is enough that $x_k^2/a_k < 1$) the function $g_k(x, \lambda) = \sum_{i \neq k} x_i^2/a_i(a_i - \lambda)$ is strictly increasing as a function of λ on the interval (a_{k-1}, a_{k+1}). Construct the following sets:

$$\begin{aligned} X_k &= \{x \in S_k^{n-1} \mid g_k(x, a_k) > 0\}, \\ Y_k &= \{x \in S_k^{n-1} \mid g_k(x, a_k) < 0\}, \\ Z_k &= \{x \in S_k^{n-1} \mid g_k(x, a_k) = 0\}. \end{aligned} \tag{25}$$

If $x^0 \in X_k$ then for any point $x = (x_i) \in S^n$ near to x_0, we have that $g_k(x, a_k) \simeq g_k(x_0, a_k) > 0$. Hence if $x_k \neq 0$ and is near to 0, there is a number λ_k greater than a_k and near to a_k, which satisfies Eq. (24), and therefore $\lambda_k = \lambda_k(x)$. It means that the function λ_k is noncritical at x and λ_k tends to a_k when x tends to x^0. Thus from Eq. (24) it follows that λ_k is extendable smoothly on X_k so that on X_k it attains the minimal value a_k, and X_k is a nondegenerate critical submanifold for λ_k. Analogously, λ_k attains the maximal value on Y_{k+1} and Y_{k+1} is a nondegenerate critical submanifold for λ_k.

Consider λ_1. By Eq. (3) we have that $X_1 = S_1^{n-1}$, Y_2 is a disjoint union of two $(n-1)$-dimensional balls, Z_2 is the boundary of Y_2 and is a disjoint union of two $(n-2)$-dimensional spheres. Consider the 2-sheet covering $S^{n-1} \times T^1 \to S^n (T^1 = S^1)$, with the ramification in $Z_2 = 2\{pt\} \times S^{n-2}$. We can lift all the functions λ_i on $S^{n-1} \times T^1$. Then λ_1 is a function depending only on the T^1 factor. Remember that on $S_1^{n-1} = \{x \in S^n, x_1 = 0\}$ we have that $\lambda_1 = a_1$, and $\lambda_2, \ldots, \lambda_{n+1}$ are the system of elliptic coordinates for the ellipsoid S_1^{n-1}. For every value of λ_1 between a_1 and a_2, $a_1 < \lambda_1 < a_2$, the ellipsoid

$$\begin{aligned} S_\pm^{n-1}(\lambda_1) = \{x \in S^n \mid \lambda_1(x) = \lambda_1, x_1 \lesseqgtr 0\} &= \left\{x \in S^n \,\middle|\, \sum_1^{n+1} \frac{x_i^2}{a_i - \lambda_i} = 1, x_1 \lesseqgtr 0\right\} \\ &= \left\{x = (x_i) \,\middle|\, \sum_2^{n+1} x_i^2 \frac{a_i - a_1}{a_i(a_i - \lambda_1)} = 1, x_1 = \pm\sqrt{a_1\left(1 - \sum_2^{n+1} \frac{x_i^2}{a_i}\right)}\right\} \end{aligned}$$

can be deformed to

$$S'_{\lambda_1} = \left\{y = (y_2, \ldots, y_{n+1}) \,\middle|\, \sum \frac{y_i}{a_i - \lambda_1} = 1\right\},$$

which is confocal to S_1^{n-1}, by putting $y_i = x_i\sqrt{(a_i - a_1)/a_i}$. By replacing λ_1 by λ_k, $k \geq 2$, one obtains that S'_{λ_1} has $(\lambda_k - \lambda_1)$ as its system of elliptic coordinates. It follows that there is a canonical diffeomorphism from $S_\pm^{n-1}(\lambda_1)$ to S_1^{n-1}, which preserves the functions $\lambda_2, \ldots, \lambda_n$. Hence on the direct product $S^{n-1} \times T^1$, the functions $\lambda_2, \ldots, \lambda_n$ can be considered as functions on the S^{n-1} component (namely, as elliptic coordinates on S_1^{n-1}).

Repeating the above procedure, we obtain a chain of 2-sheet coverings:

$$T^n \to S^2 \times T^{n-2} \to S^3 \times T^{n-3} \to \cdots \to S^{n-1} \times T^1 \to S^n. \tag{26}$$

On the whole we get a 2^{n-1}-sheet covering $T^n \to S^n$ wich is ramified over the set $\bigcup_{k=1}^{n+1} Z_k$ $(Z_1 = Z_{n+1} = \emptyset)$. Note that this set is, as usual, a subset of codimension 2 in

S^n, and when $n \geq 4$ it is a subvariety with singularities (when $n = 2$ or 3 it is a smooth submanifold).

Under the covering $T^n \to S^n$ constructed above, we have a natural decomposition $T^n = T_1^1 \times T_2^1 \times \cdots \times T_n^1$, and the function λ_i depends only on the ith component T_i^1.

We now lift the metric from the ellipsoid S^n to the torus T^n via the above covering. Note that every function λ_i is a Morse function on T_i^1, which has two maximal and two minimal points on T_i^1, and which is invariant under a natural $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action (which is generated by two reflections) on T_i^1. It follows that on every T_i^1 there is a regular periodic coordinate function q_i (which is unique up to a constant), such that the function $\lambda_i = \lambda_i(q_i)$ satisfies $\gamma_i(0) = a_i$ and has the differential

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}q_i} = \pm\sqrt{(-1)^i \lambda_i^{-1} \prod_{k=1}^{n+1}(a_k - \lambda_i)}. \tag{27}$$

Consequently, on T^n we have the system of coordinates (q_i). Let $4\omega_i$ be the period of q_i. Then we have $\gamma_i(0) = \gamma_i(2\omega_i) = a_i$ and $\gamma_i(\omega_i) = \gamma_i(3\omega_i) = a_{i+1}$. It is straightforward to check that the covering construction (26) is nothing else but the following chain of coverings

$$T^n \to T^n/\eta_{n-1} \to (T^n/\eta_{n-1})/\eta_{n-2} \to \cdots \to ((T^n/\eta_{n-1})\cdots)/\eta_1, \tag{28}$$

where each η_i is an involution given by the following map:

$$q_i \to 2\omega_i - q_i, \qquad q_{i+1} \to -q_{i+1}, \qquad q_k \to q_k \quad (k \neq i, i+1). \tag{29}$$

Recall (see, e.g. [AKN]) that the standard induced metric on the ellipsoid has the form

$$\mathrm{d}s^2 = \sum M_k(\lambda)\dot{\lambda}_k^2, \tag{30}$$

where $M_k(\lambda) = (\prod_{j=0, j\neq k}^{n}(\lambda_j - \lambda_k))/(\prod_{j=1}^{n+1}(a_j - \lambda_k))$, $\lambda_0 = 0$. From (27) we have $\dot{\lambda}_k^2 = (-1)^k \dot{q}_k^2 \lambda_k^{-1} \prod_{j=1}^{n+1}(a_j - \gamma_k)$, and therefore

$$\mathrm{d}s^2 = \sum_1^n N_k(q)\dot{q}_k^2, \tag{31}$$

where $N_k(q) = (-1)^{k+1}\prod_{j\neq k}(\lambda_j - \lambda_k)$.

Of course, the lifted metric on T^n will have the same form (31). This metric on T^n has singularities. Namely, it is degenerate in the preimage of the ramification set $\bigcup_{k=1}^{n} Z_k$.

The lifted geodesic flow on the (punctured) T^n (i.e. in the domain where the metric is nondegenerate) has the Hamiltonian

$$H = \sum_1^n p_k^2/N_k(q) \tag{32}$$

with $p_k = N_k(q)\dot{q}_k$. This Hamiltonian has the Stäckel type

$$H = \sum_1^n \frac{\Phi_{n,i} f_i}{\Phi} \tag{33}$$

with $\phi_{ij}(q_i) = (\lambda_j(q_j))^{i-1}$, $f_i(p_i, q_i) = (-1)^{n-i} p_i^2$.

Consequently, we have a system of commuting first integrals (F_i) for this geodesic flow, given by Eq. (5) $(F_n = H)$.

Eq. (8) in this situation has the form

$$D(\boldsymbol{F}, \lambda_i) = (-1)^{n-i} p_i^2 \quad (i = 1, \dots, n), \tag{34}$$

where

$$D(\boldsymbol{F}, \lambda) = F_1 + \lambda F_2 + \cdots + \lambda^{n-1} F_n. \tag{35}$$

Consider a level set $\{F_i = \text{const}\}$ in T^*K^n, where K^n is $\boldsymbol{T}^n$ minus the degenerate set of the metric. According to (35), when $\boldsymbol{F} = (F_i) = \text{const}$, $D(\boldsymbol{F}, \lambda)$ is a polynomial of degree $(n-1)$ in λ. As in the previous section, it is enough to consider the isoenergy level $H = 1$, i.e. we can fix the highest degree coefficient in the polynomial $D(\boldsymbol{F}, \lambda)$. Suppose that a level set $\{\boldsymbol{F} = \text{const}\}$ is nonempty. Then this set is given by the following system of equations and inequalities:

$$\{a_i \le \lambda_i \le a_{i+1}, (-1)^{n-1} D(\boldsymbol{F}, \lambda_i) \ge 0, p_i = \pm\sqrt{(-1)^{n-i} D(\boldsymbol{F}, \lambda_i)}\}. \tag{36}$$

Since $\lambda_i \le \lambda_{i+1}$ and $(-1)^{n-i} D(\boldsymbol{F}, \lambda_i) \ge 0$, it follows that in case $\{\boldsymbol{F} = \text{const}\}$ is nonempty we have that the polynomial $D(\boldsymbol{F}, \lambda)$ has $(n-1)$ real roots, denoted $I_1, \dots, I_{n-1}$, with the property that

$$\lambda_1 \le I_1 \le \lambda_2 \le I_2 \le \cdots \le I_{n-1} \le \lambda_n \tag{37}$$

for any point $(p_i, q_i) \in \{\boldsymbol{F} = \text{const}\}$ with $\lambda_i(q_i) = \lambda_i$.

Since $a_i \le \lambda_i \le a_{i+1}$, from (37) we have

$$a_i \le I_i \le a_{i+2} \quad (i = 1, \dots, n-1) \tag{38}$$

and

$$I_1 \le I_2 \le \cdots \le I_{n-1}. \tag{39}$$

It is clear that $I_1, \dots, I_{n-1}$ are determined uniquely by $F_1, \dots, F_{n-1}, F_n = H = 1$, so they are also commuting first integrals of our Hamiltonian system. Hence instead of taking the system of first integrals $(F_1, \dots, F_{n-1})$, we can take $I = (I_1, \dots, I_{n-1})$.

Lemma 5. *If a value $I^0 = (I_1^0, \dots, I_{n-1}^0)$ satisfies the conditions* (38) *and* (39), *then the level set $\{I = I^0\}$ is nonempty. If $I_i^0 \ne a_s$ and $I_i^0 \ne I_{i+1}^0$ for any i, s, then this level set is regular (i.e. it is a union of some Liouville tori in T^*K^n). If there are k pairs of indices (i, s) such that $I_i^0 = a_s$ and l indices i such that $I_i^0 = I_{i+1}^0 \in \mathbb{R}\backslash\{a_1, \dots, a_{n+1}\}$, then it corresponds to a codimension $(k+l)$ singularity.*

Proof. Since $I_1^0, \dots, I_{n-1}^0$ are roots of the polynomial $D(\boldsymbol{F}^0, \lambda)$, we have

$$(-1)^{n-1} D(\boldsymbol{F}^0, \lambda) \ge 0$$

if λ belongs to $[I_{i-1}^0, I_i^0]$ ($i = 1, \ldots, n-1$; $I_0^0 = -\infty$, $I_n^0 = +\infty$). From the conditions (38), (39) it follows that the following closed intervals are nonempty:

$$L_i = [I_{i-1}^0, I_i^0] \cap [a_i, a_{i+1}] \neq \emptyset. \tag{40}$$

Consequently, there are numbers $\lambda_1, \ldots, \lambda_n$ with the property $\lambda_i \in L_i, \lambda_1 < \lambda_2 < \cdots < \lambda_n$. Hence the level set $\{I = I^0\}$ is nonempty, according to (36).

Under the additional condition that $I_i^0 \neq a_s$ for any i, s, the intervals L_i are pairwise disjoint. Consequently, in this case the level set $\{I = I^0\}$ is the direct product of the curves on $T^*T_i^1(p_i, q_i)$, which are given by the equations $p_i^2 = (-1)^{n-i} D(\boldsymbol{F}^0, \lambda_i), \lambda_i \in L_i$. If $I_i^0 \neq I_{i+1}^0 (i = 1, \ldots, n-2)$, then these curves are smooth, and the level set is regular. If there is a unique index i such that $I_i^0 = I_{i+1}^0$, then L_{i+1} is just one point, and the curve $p_{i+1}^2 = (-1)^{n-i-1} D(\boldsymbol{F}^0, \lambda_{i+1})$ consists of only 4 points (because there are 4 different values of q_{i+1} with $\lambda_{i+1}(q_{i+1}) = I_{i+1}^0$). It is easy to see that in this case we have an elliptic codimension 1 singularity.

When there is a unique pair of indices (i, s) such that $I_i^0 = a_s$, and $I_j^0 \neq I_{j+1}^0$ for any j, the curves $\{p_s^2 = (-1)^{n-s} D(\boldsymbol{F}^0, \lambda_s)\}$ (in $T^*T_s^1(p_s, q_s)$, if $s \leq n$) and $\{p_{s-1}^2 = (-1)^{n-s+1} D(\boldsymbol{F}^0, \lambda_{s-1})\}$ (in $T^*T_{s-1}^1(p_{s-1}, q_{s-1})$, if $s \geq 2$) are not regular curves. They are either discrete points or have the form of a union of two circles which intersect each other transversally at two points. This situation corresponds to a stable codimension 1 singularity: the topological type of the above singular curve does not change when we perturb the value of the integrals $I_k = I_k^0$ $(k \neq i)$ (under a small change of the value I_i^0 we get a regular level set).

Thus we have proved Lemma 5 for the case $k + l = 1$. The case $k + l > 1$ can be treated similarly. □

Proof of Theorem 2. Note that the image of regular Liouville tori of the system on T^*K^n under the induced projection $T^*K^n \to T^*S^n$ are regular invariant tori for the geodesic flow on S^n, and the closer of the image of singularities in T^*K^n is singularities of the same codimension on S^n. Hence the strong nondegeneracy of the geodesic flow on S^n follows from Lemma 5.

For assertion (a), we just have to list all the codimension 1 singularities using the above lemma. The list consists of five cases:

(1) $I_i^0 = I_{i+1}^0$, $I_j^0 \neq I_{j+1}^0 (j \neq i)$ and $I_j^0 \neq a_s$ for any j, s. As was shown in the previous lemma, in this case we have elliptic singularities (i.e. type $\mathcal{A}$).

(2) $I_j^0 \neq I_{j+1}^0$ for any j, $I_i^0 = a_i$ or $I_i^0 = a_{i+2}$, and $I_j^0 \neq a_k$ for any other pair (j, k). In this case L_i is a point or L_{i+1} is a point. Again we obtain an elliptic singularity.

In cases (3)–(5), $I_i^0 = a_{i+1}$, $I_j^0 \neq a_k$ for any other pair (j, k), $I_j^0 \neq I_{j+1}^0$ for any j.

(3) $I_{i+1}^0 > a_{i+2}$, $I_{i-1}^0 < a_i$. In this case we have a codimension 1 singularity of type $\mathcal{C}_2$. (See [PZ] for the case $n = 2$. The case $n > 2$ is the same topological.)

(4) $(I_{i+1}^0 > a_{i+2}, I_{i-1}^0 > a_i)$ or $(I^{i+1} < a_{i+2}, I_{i-1} < a_i)$. In this case we have a codimension 1 singularity of type $\mathcal{B}$. It can be seen by checking the number of Liouville tori before and after a bifurcation.

(5) $(I^0_{i+1} < a_{i+2}, I^0_{i-1} > a_i)$. In this case we have a codimension 1 singularity of type $\mathcal{A}^*$.

When $n \geq 4$, all the above five cases do occur. When $n = 3$ only cases (1)–(4) can happen, and when $n = 2$ only cases (2) and (3) can happen, because of the range of the indices. Assertion (a) is proved.

As we have shown, singularities of our flow occur where some of the values I^0_i of the first integrals coincide or equal to the constants a_s. From this follows that higher-codimension singularities can be decomposed into the direct product of multipliers, and a natural list of all the multipliers consists of the following six cases:

(1) $I^0_i = a_{i+1}, I^0_{i+1} = a_{i+2}, \dots, I^0_{i+k-1} = a_{i+k}, I^0_{i+k} > a_{i+k+1}, I^0_{i-1} < a_i$. We denote this singularity by El^1_k $(k \geq 1)$.

(2) $I^0_i = a_{i+1}, I^0_{i+1} = a_{i+2}, \dots, I^0_{i+k-1} = a_{i+k}, (I^0_{i+k} - a_{i+k+1})(I^0_{i-1} - a_i) > 0$. We denote this singularity by El^2_k $(k \geq 1)$.

(3) $I^0_i = a_{i+1}, I^0_{i+1} = a_{i+2}, \dots, I^0_{i+k} = a_{i+k+1}, I^0_{i+k+1} < a_{i+k+2}, I^0_{i-1} > a_i$. We denote this singularity by El^3_{k+1} $(k \geq 1)$.

(4) $I^0_i = a_{i+1}, I^0_{i+1} = a_{i+2}, \dots, I^0_{i+k-1} = a_{i+k}; I^0_{i+k} = I^0_{i+k+1} = a_{i+k+1}, I^0_{i-1} < a_i$ or $I^0_{i-1} = I^0_{i-2} = a_i, I^0_{i+k} > a_{i+k+1}$. We denote this singularity by El^4_{k+2} $(k \geq 0)$.

(5) $I^0_i = a_{i+1}, I^0_{i+1} = a_{i+2}, \dots, I^0_{i+k-1} = a_{i+k}; I^0_{i+k} = I^0_{i+k+1} = a_{i+k+1}, I^0_{i-1} > a_i$ or $I^0_{i-1} = I^0_{i-2} = a_i, I^0_{i+k} < a_{i+k+1}$. We denote this singularity by El^5_{k+2} $(k \geq 0)$.

(6) $I^0_i = a_{i+1}, I^0_{i+1} = a_{i+2}, \dots, I^0_{i+k-1} = a_{i+k}; I^0_{i+k} = I^0_{i+k+1} = a_{i+k+1}, I^0_{i-1} = I^0_{i-2} = a_i$. We denote this singularity by El^6_{k+4} $(k \geq 0)$.

It turns out that we have the following topological equivalences:

$$El^4_{k+2} = El^2_{k+1} \times \mathcal{A}, \quad El^5_{k+2} = El^3_{k+1} \times \mathcal{A}, \quad El^6_{k+4} = El^3_{k+2} \times \mathcal{A} \times \mathcal{A}. \tag{41}$$

Hence we can restrict our attention to the first three cases of the list. El^1_1, El^2_1 and El^3_1 are codimension 1 singularities, and they were already listed in the previous list: $El^1_1 = \mathcal{C}_2$, $El^2_1 = \mathcal{B}$, $El^3_1 = \mathcal{A}^*$.

Thus we have proved assertion (b). The proof of the rest of Theorem 2 is straightforward. For example, stable closed geodesics correspond to elliptic codimension $(n-1)$ singularities, and they are given by the following equations, which are equivalent to Eq. (3): $I_1 = a_1, I_2 = a_2, \dots, I_{k-1} = a_{k-1}, I_k = a_{k+2}, I_{k+1} = a_{k+3}, \dots, I_{n-1} = a_{n+1}$ $(k = 1, \dots, n)$. □

We now replace the functions $\lambda_1, \dots, \lambda_n$ by some new functions $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n$, with the following properties (for each $i = 1, \dots, n$):

(a) $\tilde{\lambda}_i$ is a Morse function on T^1_i.

(b) $\tilde{\lambda}_i(q_i) = \tilde{\lambda}_i(-q_i) = \tilde{\lambda}_i(2\omega_i - q_i)$, i.e. $\tilde{\lambda}_i$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-invariant.

(c)
$$\frac{\mathrm{d}^k\tilde{\lambda}_i}{\mathrm{d}q_i^k}(m\omega_i) = \frac{\mathrm{d}^k\lambda_i}{\mathrm{d}q_i^k}(m\omega_i) \quad (m = 0, 1, 2, 3; k = 0, 1, 2, \dots),$$

i.e. at four points $m\omega_i$ the functions $\tilde{\lambda}_i$ and λ_i are formally the same. (Recall that $4\omega_i$ is the period of q_i in T^1_i.)

(d) Like λ_i, the function $\tilde{\lambda}_i$ attains the value a_i only at $0, 2\omega_i$ and the value a_{i+1} only at $\omega_i, 3\omega_i$.

We can then replace (γ_i) by $(\tilde{\gamma}_i)$ in all of the previous formulas. More precisely, formulas (30), (26) with (γ_i) replaced by $(\tilde{\gamma}_i)$ will give a new metric on the punctured torus K^n. Condition (d) assures the regularity of this metric in K^n. From conditions (b) and (c), it follows that this metric corresponds to some smooth metric on sphere S^n. It is clear that this new metric is also integrable. Condition (a) implies that all singularities of its geodesic flow are also nondegenerate.

In case each function by $(\tilde{\gamma}_i)$ is monotone on $(0, \omega_i)$, the corresponding geodesic flow will be topologically equivalent to that one given by the standard metric on the ellipsoid. But if these functions have critical points other than $m\omega_i$, then we will obtain new integrable geodesic flows which are not topologically equivalent to the classical case. In particular, for these new metrics, besides the singularities listed in Theorem 2 there can be nondegenerate codimension 1 singularities of the type $\mathcal{V}_k$ (also as a multiplier for higher codimension singularities). The appearance of these singularities can be seen from Theorem 1. (See [PZ] for the case $n = 2$.)

4. Topological structure of singularities El_j^i

We can compute the singularities El_j^i by induction, starting from $j = 1$. To simplify the task, we use the main result of [Z2], to know a priori that El_j^i, after a finite covering, are direct product of codimension 1 singularities. As a result, we have:

$$\begin{aligned} El_j^1 &= \mathcal{C}_2^{(1)} \times \mathcal{C}_2^{(2)} \times \cdots \times \mathcal{C}_2^{(j)}/(\mathbb{Z}_2)^{j-1}, \\ El_j^2 &= \mathcal{B}^{(1)} \times \mathcal{C}_2^{(2)} \times \mathcal{C}_2^{(3)} \times \cdots \times \mathcal{C}_2^{(j-1)} \times \mathcal{C}_2^{(j)}/(\mathbb{Z}_2)^{j-1}, \\ El_j^3 &= \mathcal{B}^{(1)} \times \mathcal{C}_2^{(2)} \times \mathcal{C}_2^{(3)} \times \cdots \times \mathcal{C}_2^{(j-1)} \times \mathcal{B}^{(j)}/(\mathbb{Z}_2)^{j-1} \end{aligned} \tag{42}$$

(upper numbers are simply enumerations).

Here the action of $(\mathbb{Z}_2)^{j-1}$ is defined as follows: each $\mathbb{Z}_2^{(i)}$ acts only on the ith and $(i+1)$th components of the product, leaving the other invariant. More precisely, recall that $\mathcal{C}_2$ has two different commuting involutions, which preserve the foliation and interchange the critical points. $\mathcal{B}$ has (only) one (orientation-preserving) involution. Denote by $b_s^{(i)}$ ($s = 0, 1$ if $i \neq 1, j, s = 0$ if $i = 1, s = 1$ if $i = j$) the above involutions on the ith component of the product. Then $\mathbb{Z}_2^{(i)}$ acts on the product of singularities by the following involution:

$$(x_1, \ldots, x_j) \to (x_1, \ldots, x_{i-1}, b_0^{(i)}(x_i), b_1^{(i+1)}(x_{i+1}), x_{i+2}, \ldots, x_j).$$

4.1. Some concluding remarks

There is another way to study the topology of geodesic flows on ellipsoids in the literature. First, following Moser, one shows that the geodesic flow can be written in the Lax form by using an extended flow called the "line flow" (see, e.g. [Mo]). Then one can linearize the flow by using the Jacobian varieties of the spectral curves (cf. [AM,Gr]). A general program by Audin states that one can study the topology of integrable systems by using

Jacobian varieties, and in a sense it was done for the geodesic flows on ellipsoids in [Au]. For example, the number of disjoint Liuoville tori over a point in the image of the moment map was computed there. (This number can also be seen from (42).) I think that this algebraic approach would lead to the same description of singularities as above, via the study of families of real algebraic curves (ovals).

This work is a generalization of our joint work with Polyakova and Selivanova [PZ,PSZ]. The idea of using ramified coverings $T^n \to S^n$ in considering geodesic flows on ellipsoids is taken from there. In this work we concentrated on the topological structure of singularities. But from this analysis one can derive other topological invariants comparatively easily. One may observe that for all integrable geodesic flows on S^n discussed above, the orbit space of the corresponding singular Lagrangian foliation is a stratified manifold (with a natural stratification by codimension of singularities), whose strata are all contractible.

An integrable system closely related to geodesic flows on ellipsoids is the Neumann problem (cf. [Mo]). In a sense, the geodesic flow can be considered as a subsystem of the Neumann problem. A nondegenerate hyperbolic singularity of maximal codimension for the Neuman problem was detected in [Dv]. We hope to return to this problem soon.

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